A primer on constant product matching markets

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Abstract

This preliminary draft presents the functioning of constant product matching markets.

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1 Introduction

Constant product matching markets are financial exchanges which match bids and asks with a new class of investors, liquidity providers (LPs). LPs act as market makers. They deposit a pair of tokens of equivalent amount in a dedicated pool and accept to take the other side of the trades according to a fixed pricing rule based on a constant product formula (see below). The platform charges a fee on trades redistributed to LPs based on their share of the pool.

Uniswap, an open-source automated protocol on Ethereum, first introduced the concept in 2018. In less than two years, the protocol has supported over $20bn transactions across 8,484 unique assets, and secured over $1bn liquidity, earning $56m fees in the process. The smart contract based protocol is decentralized and permissionless. Anyone can swap tokens, add tokens to a pool, or list a token without asking permission or needing to trust a centralized authority.

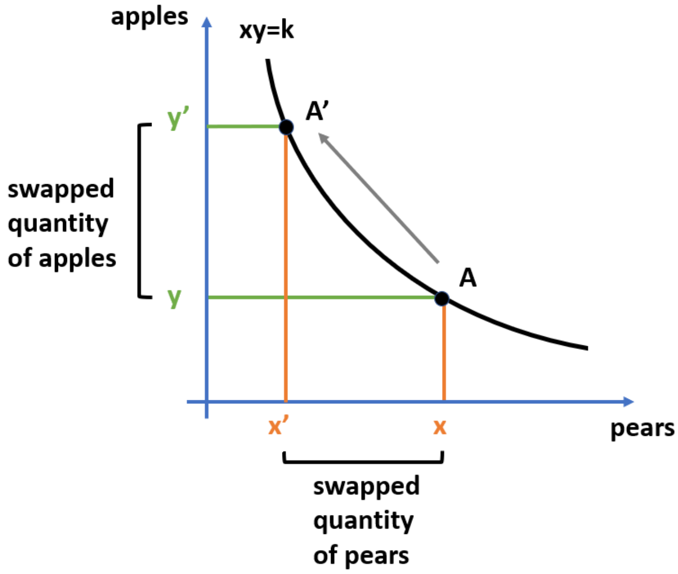
This is an intermediate level presentation of constant product market matching. The mathematics are kept as simple as possible and replaced by graphical exposition whenever feasible. The article first describes the protocol (Section II), then focuses on what investors are actually doing when they provide liquidity and what kind of risks they take (Section III). Section IV describes how volume and liquidity interact at the equilibrium and Section V concludes.

2 The protocol

Instead of matching bids and asks as in a traditional order book, a constant product exchange allows both sides of trades to instantly find a counter-party by tapping into a pre-funded liquidity pool. A distinctive aspect of the protocol is the pricing rule. Let us suppose that the protocol allows the exchange of apples and pears (replace by any ERC20 tokens tradeable on the Ethereum blockchain). When a user wants to exchange one fruit against the other, the smart contract queries the number of pears x and apples y available in the pool, computes the product $k = xy$ and offers to exchange any quantity Δx against Δy preserving the constraint $(x + \Delta x)(y + \Delta y) = k$, where k is fixed within the trade frame. If (x, y, k) denotes the initial protocol's state, the state transits after the trade to (x', y', k) with $x' = x + \Delta x$ and $y' = y + \Delta y$. The product k only varies when fruits are added into or withdrawn from the pool by LPs, or when transaction fees paid by traders expand the pool. In the following, we abstract of, but have to keep in mind, transaction and network fees paid by traders to LPs and Ethereum miners respectively.

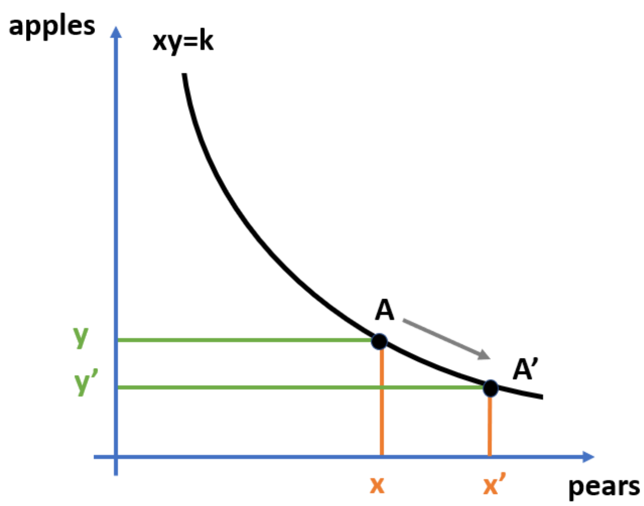
Graphically, if a trader is willing to trade the two fruits while the pool's state is A $= (x, y, k)$, he may choose any quantity $(x' - x)$ and $(y' - y)$ satisfying $x'y' = k$ in Figure 1. If the trader buys (removes from the pool) $(x' - x)$ apples, he gives up (adds up to the pool) $(y' - y)$ pears. The funds move from A to A' along the $xy = k$ curve, also called the bonding curve. Symmetrically, selling pears and buying apples would move the pool to the other direction but still

Figure 1: How the protocol moves along the constraint $xy = k$



along the bonding curve, as in Figure 2.

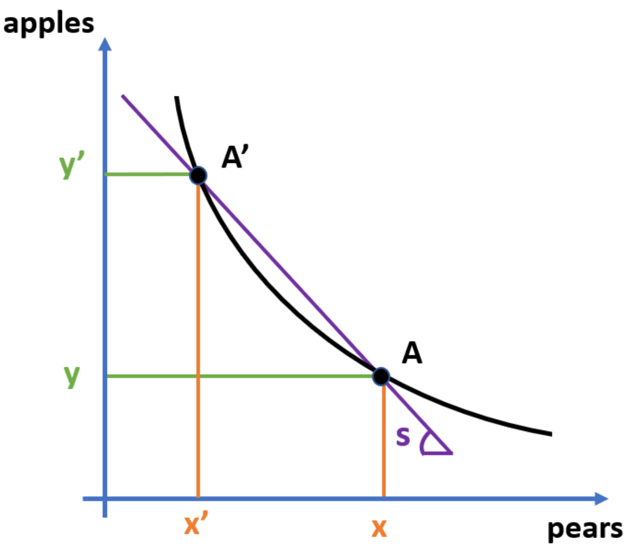
Figure 2: Buying apples and selling pears



2.1 The exchange rate

Because the protocol does not quote prices explicitly, we need to define the relative price of a traded asset pair. The exchange rate indicates how many apples a trader can obtain in exchange of one pear, or the ratio $(y' - y)/(x' - x)$. It is in Figure 3 the triangle's height $y' - y$ divided by its base $x' - x$ or the absolute slope of the purple line crossing A and A'.

Figure 3: The exchange rate



For instance, if a trader buys $x - x' = 4$ pears in exchange to $y' - y = 12$ apples, the trade's exchange rate is $12/4 = 3$. This is also the price of one pear expressed in units of apples. The steeper the slope, the more expensive pears in terms of apples.

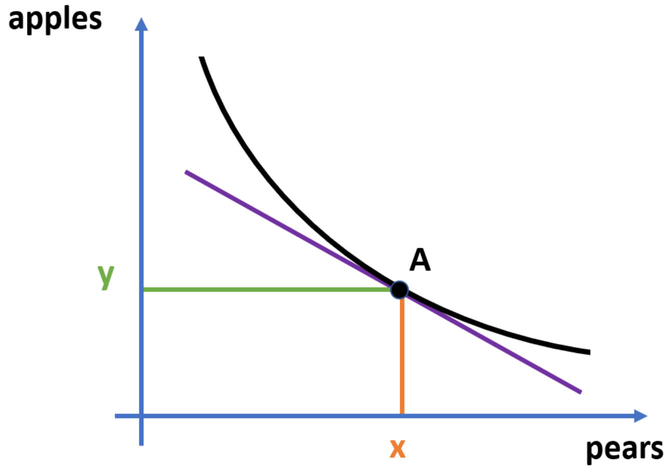
Here, apples play the role of the numeraire. Pears are valued by their capacity to buy apples. The choice of the numeraire is important as it serves to evaluate the impact of price variations on the liquidity pool. If a stablecoin is one of the two assets, it may serve as the numeraire as most investors think in dollars. The choice of the numeraire is less clear if the pair is composed of two tokens which values are not pegged to the dollar. For instance if the pair is composed of ETH and WBTC and the WBTC price falls relative to ETH, this will be edited as a loss if the numeraire is ETH and as a gain if it is WBTC. It may be best to choose conventionally one of the two as the numeraire, and then convert the value in dollars.

2.2 The market price

The same way bids and asks propose a broad spectrum of exchange rates in an order book, traders face a continuum of exchange rates on Uniswap, depending on the quantity they are willing to trade. The reference price is conceptually different from posted exchange rates. In an order book, the market price is bounded by the best bids and asks posted by sellers and buyers. A trader is guaranteed to obtain an exchange rate within this range if the quantity he sells or buys is small enough. The market price is approximately pinned down if the market is sufficiently liquid in which case the bid-ask spread is negligible.

Similarly, the market price proposed by the protocol is by definition the best exchange rate a trader can obtain by swapping an arbitrarily small quantity. Let's see what the definition graphically means. In Figure 3, when $(x' - x)$ and $(y' - y)$ approach zero, A' is getting closer and closer to A. At the limit, A' becomes visually indistinguishable from A. The exchange rate is still given by the slope of the line crossing A and A', which is tangent to the bonding curve, as in Figure 4.

Figure 4: The market price is the slope at the point of tangency



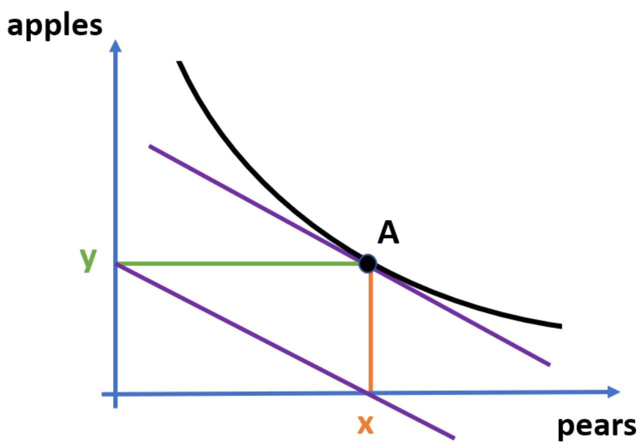
Note that if the trader were taking the other side of the trade, so that A' would approach A from below, he would benefit from the same limit exchange rate as the line crossing the two states would also become tangent to the bonding curve. This parallels a property of liquid order book markets in which investors trading small quantities face approximately the same price whether they sell or buy.

The market price p is the negative of the first derivative of $y = k/x$ computed in state A $= (x, y, k)$ and is just the ratio of the quantities of the two fruits available in the pool:

$$p = -\frac{\partial y}{\partial x} = \frac{k}{x^2} = \frac{y}{x}$$

The more pears in the pool, the lower its price. The equality $p = y/x$ is an

Figure 5: The price is the ratio of the two quantities in the pool



elegant property of constant product curves, illustrated in Figure 5. The market price $p = y/x$ is the slope of the line crossing y and x. It parallels the line tangent to the bonding curve in A. The two lines keep their parallelism when A is moving along the bonding curve.

It is easy to recover the protocol's price by dividing the quantity of apples held in the pool by the quantity of pears. The protocol is also robust to rebasing (or stock splits in traditional finance). For instance, suppose that pears are tokenized where one token is worth one pear. If its number suddenly doubles, so that one token only buys half a pear, its price is automatically divided by 2. Another nice property of the constant product formula is that the protocol always exhibits a strictly positive price whatever the quantities available in the pool as long as it starts with a positive quantity of the two assets. The trades cannot exhaust the stock of one asset as its price is increasing sufficiently fast. The fact that the price strongly reacts to changes in available quantities has however some downsides reviewed later on (see sub-sections on slippage costs and impermanent losses).

2.3 Adherence to equilibrium price

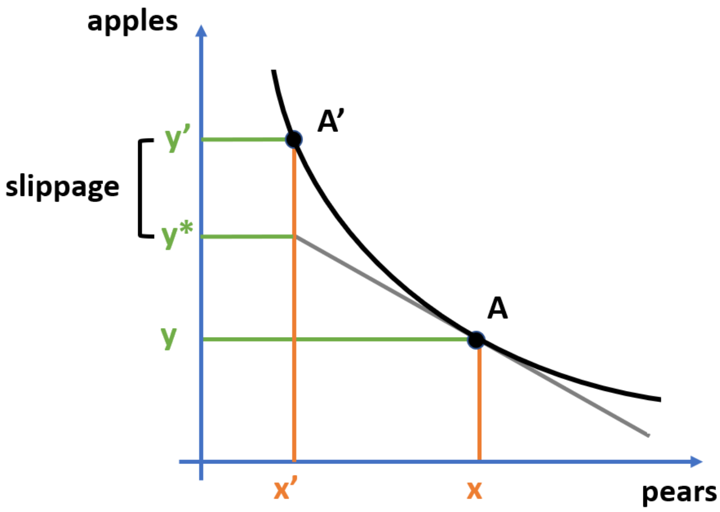
The exchange takes as given the equilibrium price p determined by demand and supply at the whole market level. However, the equality $y = px$ is not a constraint that the protocol tries to meet (it has no clue about what is the market price), but a consequence of market forces and arbitrage. If the exchange rate deviates from the market price, arbitrageurs buy the cheapest asset from the protocol and sell it in other exchanges at a premium. In doing so, arbitrageurs bring the exchange rate in the close neighborhood of the market price where arbitrages are no more profitable given trade costs and network fees. By driving the protocol's exchange rate close enough to the equilibrium price, arbitrageurs make a profit at the expense of LPs, whose loss is called "impermanent loss" (see below).

The fact that the exchange rate stays close to the equilibrium price means that the pool's quantity ratio y/x is pinned down by the market. The cheaper pears compared to apples (the lower p), the larger the quantity of pears accumulated in the pool compared to the quantity of apples. This means that the protocol automatically follows a counter-cyclical investment policy. It buys and accumulates the asset which price is down and sells the asset which price is up. If one of the two asset asset is a stablecoin, it gradually cashes out when the price of the other asset is trending up. This strategy is profitable if the price tends to revert to its mean eventually. It is inefficient if the price is trending downward (upward) and never recovers (falls back). It is disastrous if it goes to zero, in which case the rotten fruit completely drives out the good one in the pool.

2.4 Price slippage

The price slippage is the difference between the effective exchange rate obtained by the trader and the market price. In absence of slippage, the trader would pay the market price regardless of the volume traded. In Figure 6, the trader would exchange the smaller quantity of apples y^* instead of y' against $x' - x$ pears .

Figure 6: Price slippage



The larger the trade, the further away A' from A along the curve, and the most unfavorable the exchange rate. Price slippage is a component of transaction costs, which also include exchange trading fees and gas fees for using the Ethereum Network.

To compute slippage costs, suppose that, starting from the state (x, y, k), a trader is willing to buy Δx pears, thereby moving the pool's quantity to $x' = x - \Delta x$. The number of apples the trader will have to spend is (with k replaced by xy):

$$\Delta y = \frac{xy}{x - \Delta x} - y = y\frac{\Delta x}{x - \Delta x} = p\frac{x\Delta x}{x - \Delta x}$$

Without price slippage, the trader would pay $\Delta^* y = p\Delta x$. The loss rate is therefore:

$$L = \frac{\Delta y}{\Delta^* y} - 1 = \frac{\Delta x}{x - \Delta x}$$

The larger the pool's size x, the smaller the loss rate. Moreover, the larger the trade's size the higher the loss:

$$\frac{\partial L}{\partial \Delta x} = \frac{\Delta x}{(x - \Delta x)^2} > 0$$

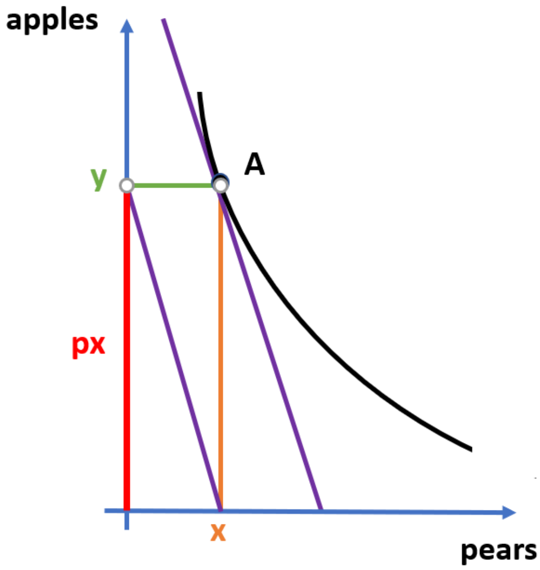
This result highlights the scale economies associated with a liquidity pool. The more abundant the liquidity, the less price slippage for traders. This attracts additional trade volumes, especially large trades which suffer the most from slippage, and ultimately generates more fees for the LPs.

2.5 Pool's market value

The market value of the assets held in the pool is the quantities expressed in terms of the numeraire (here apples):

$$V = px + y$$

Figure 7: The protocol always holds an equal share of the two assets ($y = px$)



Since the protocol sticks to the constraint $px = y$, regardless of market price p, the pool is always invested 50:50 in the two assets. This is a remarkable rule which is illustrated in Figure 7. The vertical red line is both the value y of apples and the value px of pears in the pool. This is due to the fact that the slope of the purple line crossing y and x is also the market price p.

The market value of the two assets in the pool can be expressed in terms of p and k from the two constraints $xy = k$ and $p = y/x$. We obtain $y = \sqrt{kp}$ and $x = \sqrt{k/p}$ and therefore

$$V = px + y = 2\sqrt{kp}$$

Instead of evolving linearly with the market price, the pool's market value is a function of its square root. This has fundamental implications for the return earned by LPs, which are now analyzed.

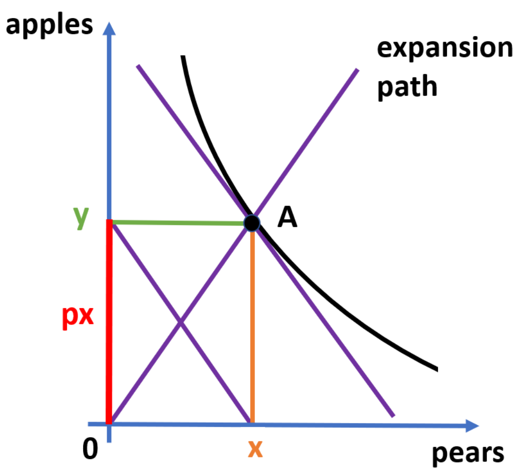
3 Protocol's dynamics

I review the consequences for the protocol of a change in liquidity, a permanent price variation and a temporary price variation.

3.1 Effects of a change in pool's liquidity

When a LP adds liquidity to to the pool, or withdraws from it, or when collected fees from trades grow the pool, the operation should preferably not alter the market price. First, because liquidity variations do not signal any change in the relative scarcity of the two assets. Second, introducing noise in the price offers a free lunch to arbitrageurs.

Figure 8: Expanding or shrinking the pool without changing the price



In Figure 8, given market price p, the pool may shrink or expand along the expansion path of the liquidity pool. Because the slope of the expansion path is the negative of the slope of the diagonal $y : x$, which is also the market price

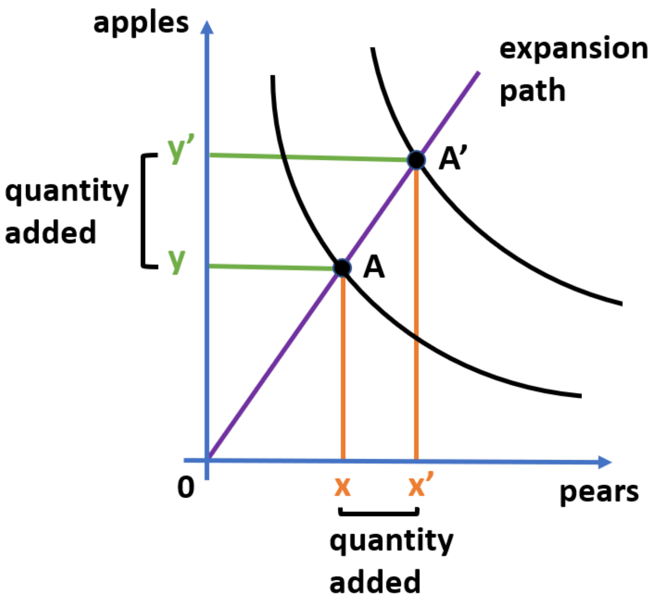
$p = y/x$, moving the protocol's state along the expansion path does not alter the market price.

If the state of the protocol is (x, y, k), any addition or withdrawal of liquidity should be a fraction α of the two quantities: $\Delta x = \alpha x$ and $\Delta y = \alpha y$, such that the market price stays the same:

$$p = \frac{y + \Delta y}{x + \Delta x} = \frac{(1+\alpha)y}{(1+\alpha)x} = \frac{y}{x}$$

The constant k is updated to $k' = (1+\alpha)^2 k$. Figure 9 shows the case of liquidity addition.

Figure 9: Expanding the pool by providing additional liquidity



3.2 One-sided liquidity provision

The last subsection presumed that the LP must provide the two assets in a given proportion and withdraw from the pool the two assets in another given proportion, depending on how the price has evolved. It may be however convenient to let the LPs choose in which proportion they provide and withdraw liquidity. This economizes on transaction costs if the LPs have to convert part of their assets before adding to the pool or after withdrawing from it.

Let us see how the protocol may handle LPs adding or withdrawing Δx pears and Δy apples in unconstrained proportion. To start, suppose that a LP adds in the pool a basket of fruits containing too many pears. Given market price p, the operation can be decomposed into a balanced two-sided liquidity provision and a one-sided addition of pears. The provision is made up of Δy apples and $\Delta y/p$ pears (the balanced part) plus $\Delta x - \Delta y/p$ pears (the one-sided addition). Since the balanced part is easily handled, we can focus on the one one-sided part denoted $\delta x = \Delta x - \Delta y/p$.

As illustrated in Figure 10, the transaction moves the protocol from A to A' and distorts the exchange rate by making pears cheaper than they actually are. Arbitrageurs or traders chasing the best exchange rate across platforms buy the extra pears until the protocol's state lands in A". The market price observed in A is restored, as indicated in Figure 11. The missing apples that the LP did not add to the pool are eventually provided by the traders and arbitrageurs.

The transit from A' to A" bears resemblance to what happens when a trade

Figure 10: Expanding the pool by providing only pears

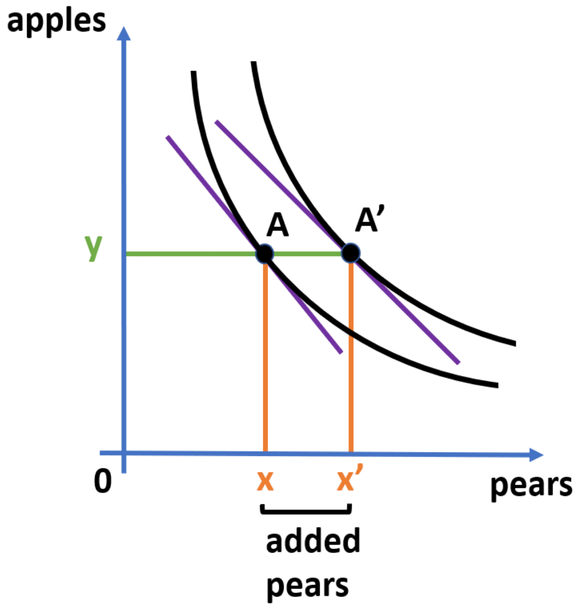
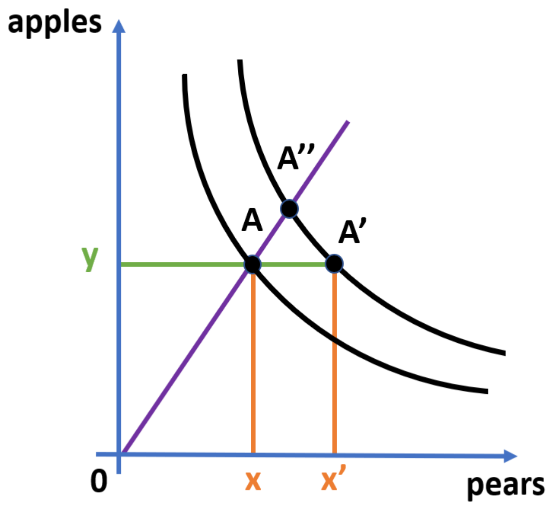


Figure 11: The surplus of pears is arbitraged away from A’ to A”



depletes the stock of apples in the pool and makes them temporarily more expensive than they actually are given the market price. In this event, the protocol first moves from A” to A’ and then back to A” thanks to the opposite trades executed by arbitrageurs and traders. Other traders profit because the first trader paid slippage costs. The pool's value is unaffected in the process. Likewise, when a LP adds one-sided assets in the pool, he should bear the transition costs for other LPs not to be harmed.

Starting from the state A = $(x, y, k = xy)$ and price $p = y/x$, the one-sided provision moves the protocol to the state A' = $((1+\delta)x, y, k' = (1+\delta)xy)$. The excess of pears is arbitraged away until the protocol lands in state A" = (x'', y'', k'') with $x'' = \sqrt{k'/p}$ and $y'' = \sqrt{k'p}$ or equivalently

$$x'' = \sqrt{\frac{(1+\delta)xy}{y/x}} = x\sqrt{1+\delta}$$

and

$$y'' = \sqrt{(1+\delta)xy(y/x)} = y\sqrt{1+\delta}$$

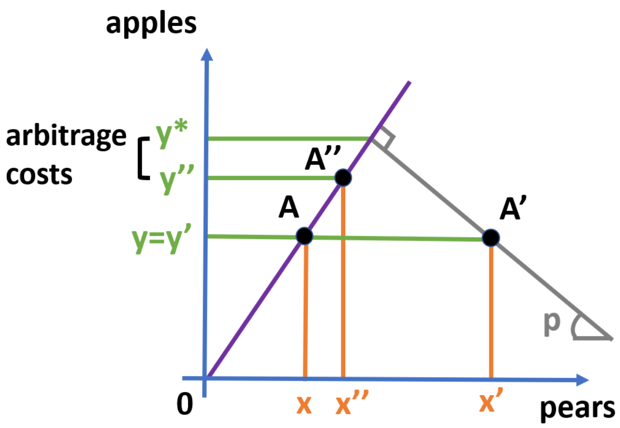
Other LPs would not be harmed by the one-sided provision if the migration from A' to A" were executed at the prevailing market price p, i.e. by exchanging $x' - x'' = (1+\delta)x - x\sqrt{1+\delta}$ pears against $y^* = p(x' - x'')$ fresh apples with y^* indicated in Figure 12. Instead, because, the protocol moves along the bonding curve, the pool ends up with only $y'' = y\sqrt{1+\delta}$ apples.

The new pool's value is $V'' = px'' + y'' = \sqrt{1+\delta}(px+y) = \sqrt{1+\delta}V$. Since the LP adds δx to the pool, the pool's market value without arbitrage costs would increase by the factor

$$\frac{p(1+\delta)x + y}{px + y} = \frac{p(1+\delta)x + px}{2px} = 1 + \frac{\delta}{2}$$

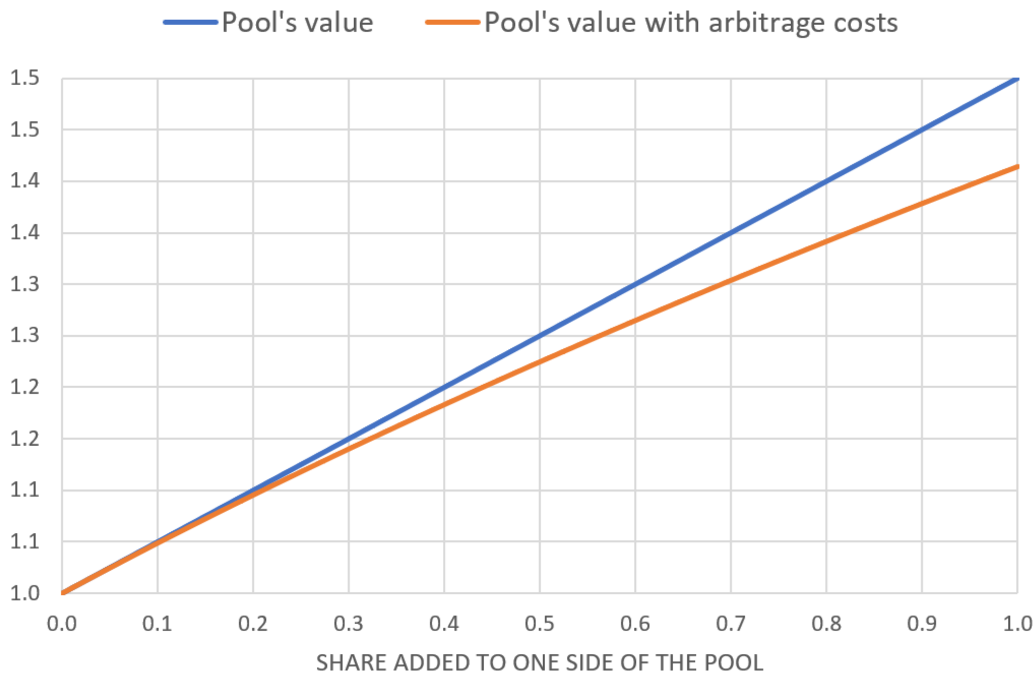
Instead, the pool's value is increased by $\sqrt{1+\delta}$ which is less than $1 + \delta/2$. Figure 13 plots the pool's market values with and without arbitrage costs, start-

Figure 12: The protocol incurs arbitrage costs during the transition



ing from $V = 1$, in function of the share δ added to the pool. As with slippage costs, the gap is small for moderate additions, then rapidly increases.

Figure 13: Pool's value with and without arbitrage costs in function of the share added to the pool



For the one-sided provision to be neutral for other LPs, the loss of value

Table 1: Loss rate from providing one asset to the pool

Share added (percent)	0.01	0.1	0.2	0.5	1	5	10	20	50
Loss rate (percent)	0.0025	0.025	0.050	0.12	0.25	1.22	2.38	4.55	10.10

should be charged to the LP at the origin of the operation. By adding $p\delta x$, the claimable pool's share is $(\sqrt{1+\delta}-1)V$. Considering the LP would claim the value $(\delta/2)V$ from the pool in absence of arbitrage costs, the loss rate is:

$$\frac{(\delta/2-\sqrt{1+\delta}+1)V}{p\delta x}=\frac{(\delta/2-\sqrt{1+\delta}+1)2}{\delta}=\frac{\delta+2(1-\sqrt{1+\delta})}{\delta}$$

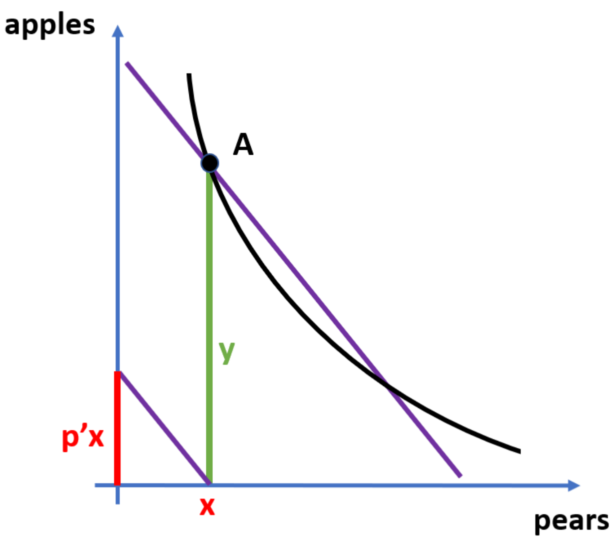
Table 1 shows the loss rate in function of the share δ added to the pool. The trade-off between provision's size and arbitrage cost resembles the one between trade's size and slippage cost. The cost is negligible when the size is small compared to the pool's size but rapidly increases for relatively medium and large additions. As with slippage costs, dividing the provision in smaller sizes may reduce the total cost.

One-sided withdrawing is handled the same way as one-sided provision. The LP who exits ends up with less wealth compared to a balanced withdrawal due to arbitrage costs. The trade-off between size and costs is similar. Overall, the LPs may add one asset and withdraw the same asset or the other one. Or they may provide a 20:80 proportion of the two assets and withdraw a 80:20 proportion. Note that this possibility given to LPs is different from a price exposure to only one of the two assets. Here the pool's share is adjusted by taking into account arbitrage costs but after this correction, the LP has a claim on the whole asset pool.

3.3 Effects of a price change

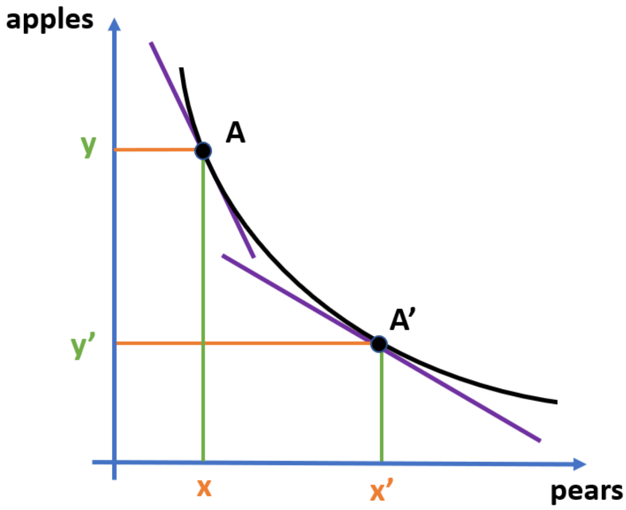
Suppose now that the price of pears drops from p to p'. The market value of pears held in the pool instantly decreases from px to $p'x$. The new value is indicated in red in Figure 14, with the two purple lines having the same slope equal to the new market price. The pool's market value $V' = p'x + y$ decreases as a result.

Figure 14: Instantaneous effect of a price decrease on the pool's value



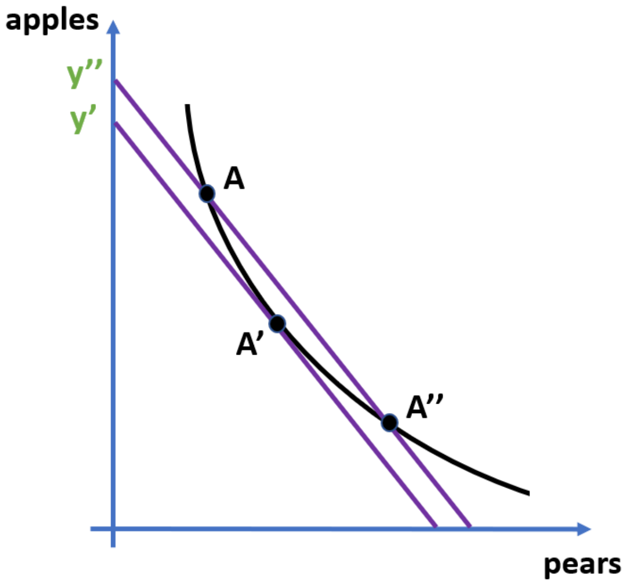
Then, rapidly, comes a composition effect as the exchange rate y/x lags above the new equilibrium price p'. Arbitrageurs buy apples against pears at a discount, depleting the stock of apples and increasing the stock of pears in the pool. The process continues until the exchange rate realigns with the equilibrium price: $y'/x' = p'$. The new protocol state is A'= (x', y', k) in Figure 15.

Figure 15: The pool mix is changed, the exchange rate realigns with the market price



The pool's value becomes $V' = 2\sqrt{kp'}$. Now suppose that the investor did not provide liquidity to the pool and just held $x = \sqrt{k/p}$ pears and $y = \sqrt{pk}$ apples in his portfolio. After the price decrease, he would still be in A in Figure 16. The line crossing A represents all the combinations of the two fruits that the investor may obtain by trading pears against apples at market price p'. The line is above the one crossing A' for all combinations. Assuming that the LP owns the entire pool (owning a fraction of it would not change the conclusion), the pool graphically underperforms compared to the investor's portfolio. In particular, the investor could "cash out" in apples and obtains the quantity y'' in Figure 16, compared to only y' if the liquidity pool were liquidated.

Figure 16: More apples and pears outside the protocol



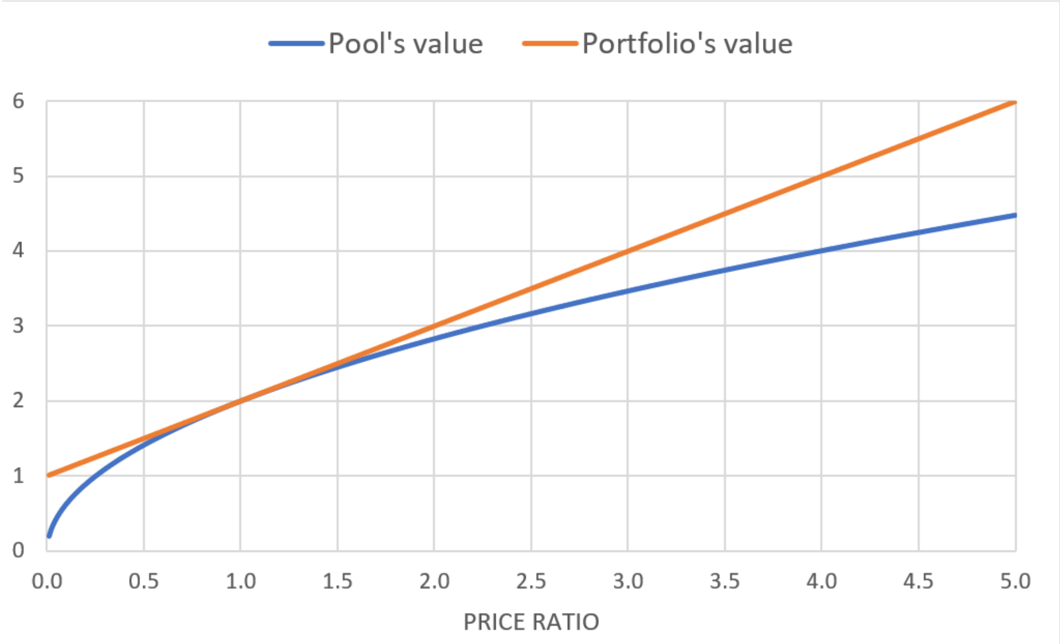
The pool also underperforms by a similar margin if the price increases. In Figure 16, the pool's state could as well start from A" instead of A and, after the price rises, moves to A' with the same loss for the LPs.

The holding investor benefits from a constant (or quasi-constant in case of price slippage) exchange rate along the portfolio line crossing A and A". To the contrary, the protocol automatically rebalances the pool at a substandard exchange rate due to adherence to the bonding curve.

The missing return can be evaluated by assuming that the investor initially creates a pool at market price p with $x = \sqrt{k/p}$ pears and $y = \sqrt{pk}$ apples in proportion $y/x = p$. If the price varies to p', the pool's value becomes $V^p = 2\sqrt{p'k}$ whereas the holding portfolio's value changes to $V^h = y + p'x = \sqrt{pk} + p'\sqrt{k/p} = (1 + p'/p)\sqrt{pk}$. The loss ratio of providing liquidity relative to holding a portfolio is given by

$$\frac{V^p}{V^h} = \frac{2\sqrt{p'/p}}{1 + p'/p} \leq 1$$

Figure 17: Pool's and portfolio's value in function of the price



The market value of the pool and the portfolio are represented in Figure 17 in function of the price ratio p'/p by assuming (without loss of generality) that both the investor and the protocol start with $x_0 = y_0 = 1$ fruits and that the initial price is $p = 1$. LPs face two risks compared to simply holding a balanced portfolio. They capture only a fraction of the gains if the price increases and they bear a disproportionate share of the loss if the price decreases. Table 2 shows loss rates $V^h/V^p - 1$ (by how much LPs' wealth is lower compared to investing outside the protocol) given various price decrease or increase rates $p'/p - 1$.

Of course, the missing return in the upside and the aggravated loss in the downside may be compensated by the fees earned during the process.

Table 2: Loss rates in function of price variation rate

Price decrease rate (percent)	-1	-5	-10	-15	-20	-30	-50	-80
Loss rate (percent)	-0.001	0.03	-0.14	-0.33	-0.62	-1.57	-5.72	-25.46
Price increase rate (percent)	5	10	20	30	50	100	200	300
Loss rate (percent)	-0.001	-0.03	-0.24	-0.41	-0.85	-2.02	-5.72	-13.40

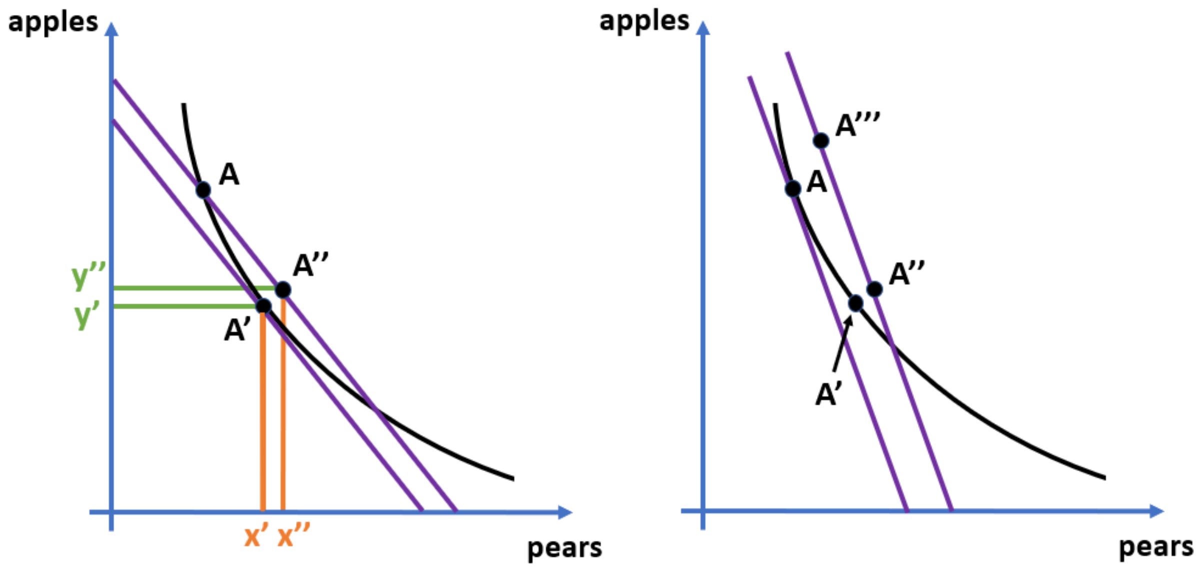
3.4 Effects of a temporary price change

Next, assume that the price varies from p to p' and, after a while, goes back to p. The protocol returns to its initial state, and the pool's value to its initial market value. Likewise, an investor holding a 50:50 portfolio outside the protocol would get the same market value back.

The fact that the two strategies yield the same result seems at first disturbing. In the pooling strategy, the pool incurs arbitrage costs twice, a first time when the exchange rate is driven to p' and a second time when it goes back to p. In the holding strategy, the investors avoids arbitrage costs altogether, yet ends up with the same final wealth. How is it possible?

The reason is that, contrary to the individual investor, the protocol rebalances the pool to a 50:50 shares of apples and pears whenever the market price changes. In doing so, it accumulates the fruit which price has decreased and decumulates the fruit which price has increased. It follows that rebalancing is favorable to portfolio's return if the price reverses to it initial level, which is assumed here. On the other hand, the protocol does not rebalance efficiently, giving away return to arbitragers for free. Overall, the operation is neutral compared to a simple holding strategy without rebalancing.

Figure 18: Better rebalancing than not



We conclude that an individual investor who would 50:50 rebalance, as the protocol does, but at a better exchange rate, would perform better. This is shown in Figure 18. In the left-hand figure, after the price change, the protocol transits from A to A'. The individual investor now rebalances his portfolio from A to A". He holds more of the two fruits than the protocol and gets ready for the price reversal. In the right-hand figure, the price reverts to its initial level and the protocol to A. The investor is not anymore in A but in A" and outperforms the protocol by an increasing margin. He then could rebalance his portfolio once again from A" to A"'.

In the left-hand figure, the superiority of the holding strategy comes from the fact that the investor rebalances along a straight line whereas the protocol performs the same operation along a curved line. Rebalancing costs are linear with the trade size in the first case and have a quadratic shape in the second case.

More formally, assume that the protocol and the investor both start with $x_0 = \sqrt{k/p}$ pears and $y_0 = \sqrt{pk}$ apples. After the price changing from p to p', the investor rebalances his portfolio by choosing x_1 and y_1 such that $(y_1 - y_0) + p'(x_1 - x_0) = 0$ and $y_1 = p'x_1$. Solving the system gives $x_1 = \frac{1}{2}\Big(1 + \frac{1}{p'} + \frac{1}{p}\Big)\sqrt{pk}$ and $y_1 = \frac{1}{2}\Big(1 + \frac{p'}{p}\Big)\sqrt{pk}$. After the price reverting to p, the portfolio' value is:

$$V^h = \frac{1}{2}\Big(2 + \frac{p'}{p} + \frac{p}{p'}\Big)\sqrt{pk}$$

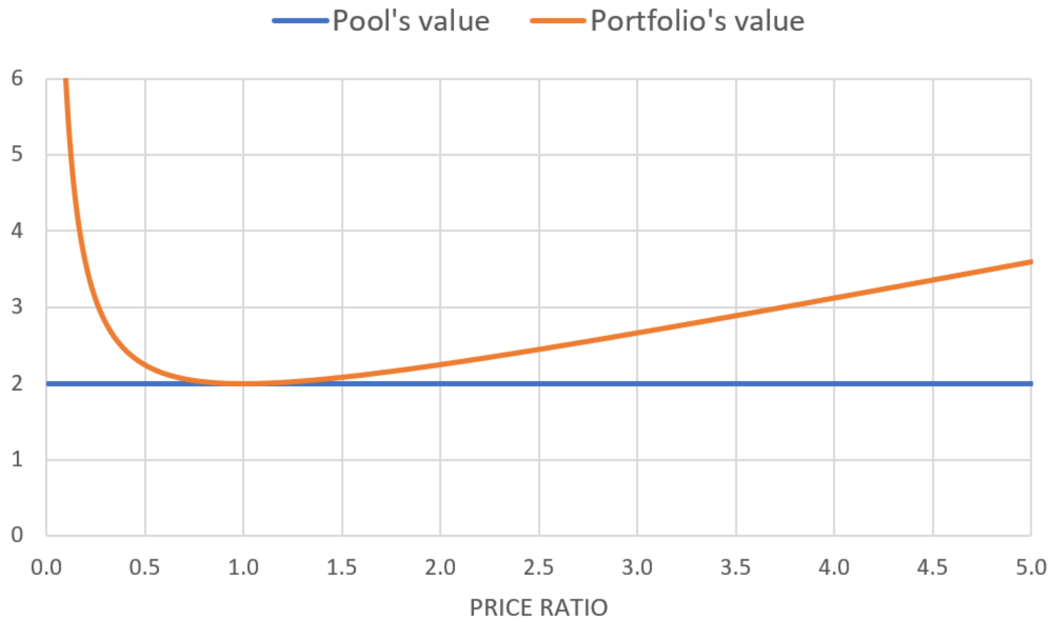
The pool's value after the price returning to p is $V^p = \sqrt{pk}$. Investor's excess return is therefore:

$$\frac{V^h}{V^p} - 1 = \frac{1}{4}\Big(2 + \frac{p'}{p} + \frac{p}{p'}\Big) - 1$$

which is positive since $V^h/V^p - 1 > 0$ implies $(p - p')^2 > 0$.

The excess return of the holding strategy with rebalancing compared to liquidity pooling can be computed by posing (without loss of generality) that both the investor and the protocol start with $x_0 = y_0 = 1$ fruits and that the initial price is $p = 1$. The pool's and portfolio's market values are plotted in Figure 19 against the price ratio p'/p which indicates by how much the price deviates from its initial value before returning to it.

Figure 19: Pool's and portfolio's value with rebalancing

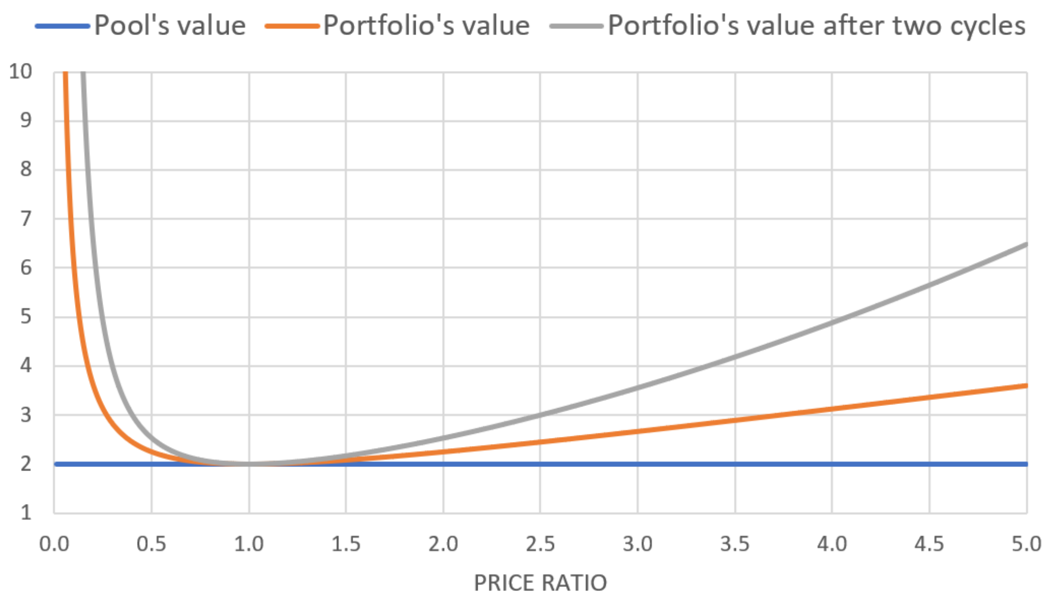


The excess return is negligible for small variations but increases at an increasing pace for larger ones. If the price deviates by -18% or +22% before returning to its initial value, excess return $V^h/V^p - 1$ is 1%. It increases to 5% if the price temporarily deviates by -36% or $+56\%$ and is up to 11% if it deviates by -50% or $+200\%$.

Note that the ability to rebalance after a price change may be hampered by transaction costs (price slippage plus transaction and network fees). The investor may not rebalance for limited price changes for this reason. On the other hand, computed losses are valid for one rebalancing. Excess return should be compounded for a sequence of price variations and rebalancing. The loss is therefore expected to expand with time.

As an illustration, Figure 20 presents under the same baseline ($x_0 = y_0 = p = 1$) the final portfolio's value after two cycles during which the price goes twice from p to p' before returning to p.

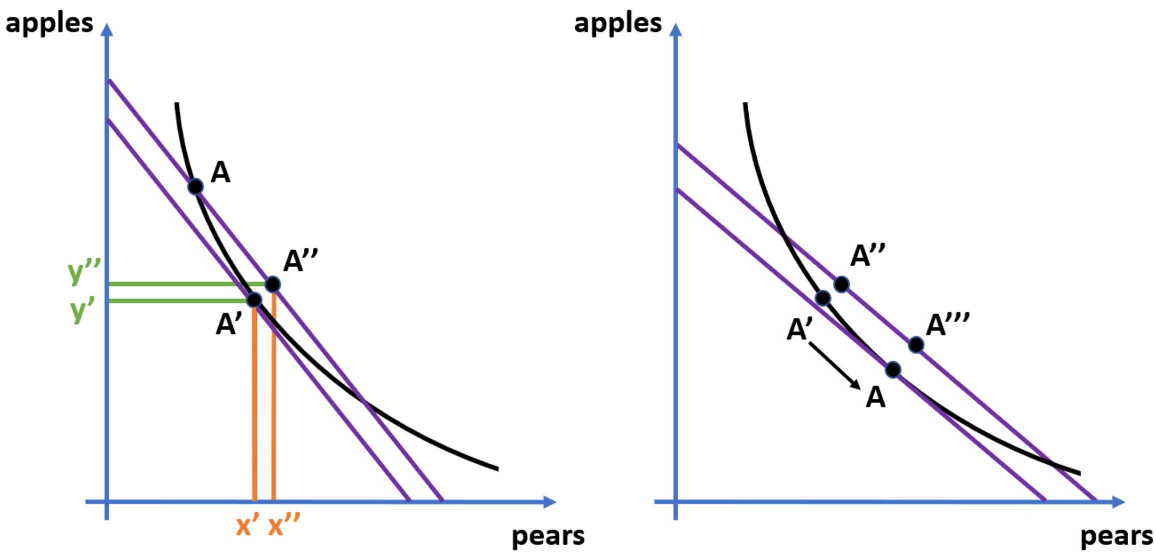
Figure 20: Pool's and portfolio's value after rebalancing twice



Rebalancing is a profitable strategy when the price reverts to its mean. Importantly, holding and rebalancing a 50:50 portfolio still dominates liquidity pooling if the price does not revert and continues to go up or down. This is illustrated in Figure 21.

The left-hand figure is the same than the previous one. The investor rebalances his portfolio from A to A". In the right-hand figure, the price decreases further. The protocol automatically rebalances from A' to A. After rebalancing from A" to A"', the investor is still better off as he owns more pears and ap-

Figure 21: Pool's and portfolio's value without price reversion



ples than the pool does. The reason is that, while both the protocol and the investor rebalance the assets, investor's rebalances at lower costs whatever the future direction of the price.

Let us take the previous scenario at the point when the investor rebalances to $x_1 = \frac{1}{2}(1 + \frac{1}{p'} + \frac{1}{p})\sqrt{pk}$ and $y_1 = \frac{1}{2}(1 + \frac{p'}{p})\sqrt{pk}$ after the price going to p'. But now, the price does not revert to p, but further increases or decreases to p''. Her portfolio value is:

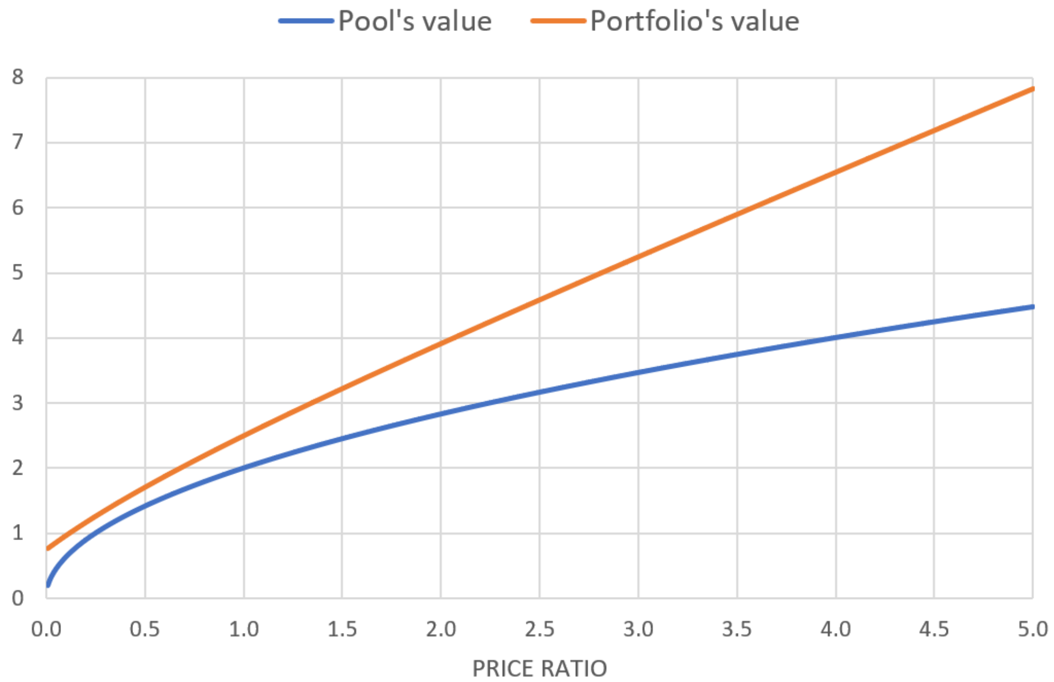
$$V^h = \frac{1}{2}\left(p'' + \frac{p''}{p'} + \frac{p''}{p} + 1 + \frac{p'}{p}\right)\sqrt{pk}$$

Under the same baseline ($x_0 = y_0 = p = 1$) plus the assumption that p' is half way between p and p'' ($p' = (p + p'')/2$), we obtain a portfolio's value

$$V^h = \frac{5p''}{4} + \frac{p''}{p'' + 1} + \frac{3}{4}$$

which can be computed for every price ratio p''/p.

Figure 22: Pool's and portfolio's value when price does not revert



The result is shown in Figure 22. Once again, the holding investor performs better than LPs.

3.5 Impermanent loss

A common view about the risk borne by LPs is that the loss from liquidity pooling is "impermanent", i.e. vanishes when the price goes back to its initial value. However, the previous analysis has shown that even in this case, LPs are at a loss compared to holding and rebalancing a portfolio. When rebalancing is allowed, even mean reverting price fluctuations are costly for LPs. Also, considering that the relative loss incurred by LPs comes from the protocol relying on arbitrageurs to rebalance at unfavorable exchange rate, quadratic rebalancing costs is a good description of the loss of return suffered by LPs.

One could argue that most investors are not willing to rebalance, or simply do not care. However, most of them do not hold a 50:50 portfolio neither, yet this strategy is commonly used as a benchmark to evaluate the risks associated with liquidity pooling. As a matter of fact, we do not need as a counterfactual a strategy that investors actually follows but one that is the closest to what the protocol performs. Since the protocol keeps a 50:50 share of the two assets, so should do the counterfactual strategy.

Another drawback of not using a rebalanced portfolio is comparing a protocol which keeps managing a 50:50 pool with a portfolio which shares drift away when the price fluctuates. For instance, a price increase of 50% raises the share of pears to 60% in absence of rebalancing. Future deviations of return result from both price variations and protocol's automatic rebalancing without the possibility to isolate the latter effect.

On the other hand, measuring quadratic rebalancing costs by computing the counterfactual return of a rebalanced portfolio is not immune from conventions. First, the portfolio should be rebalanced only after sufficiently large price variations to economize on transaction costs. Second, price slippage depends on the amount of capital rebalanced and network fees at the rebalancing date. A rule of thumb could be to rebalance the portfolio every time the price deviates by, say, more than 20%. Or to make rebalancing an infrequent occurrence by allowing a higher threshold like 50%. The higher the threshold, the more the rebalanced portfolio resembles the non-rebalanced portfolio. If the portfolio is never rebalanced, the computed loss of return is a lower bound of actual losses.

4 Market equilibrium

The exchange can be viewed as a venue where demand and supply of liquidity meet. The demand is the volume of transactions over a given time span. It depends on transaction costs which are, setting aside network fees, trade fees and price slippage. Transaction costs for a trade of size Δx sum to (see section on price slippage):

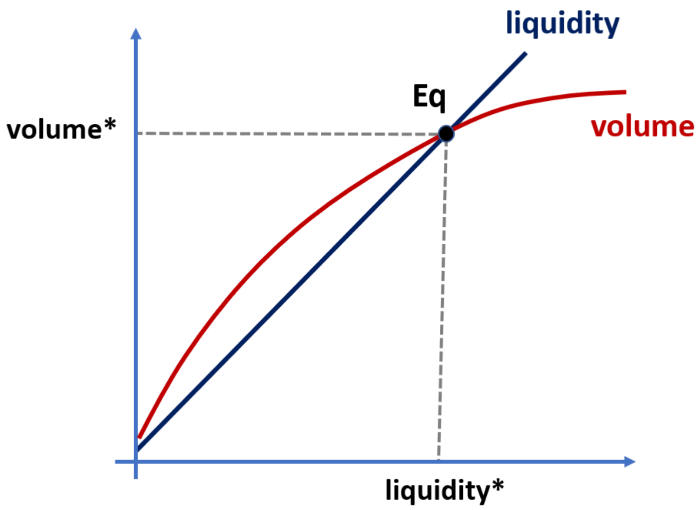
$$\tau + \frac{\Delta x}{x - \Delta x}$$

The demand for transactions depends on the fee rate τ and size of the liquidity pool x as more liquidity implies less slippage.

The supply of liquidity is the market value of the pool $V = y + px = 2px$. It depends positively on expected fees τ applied to volume vol and negatively on the risk associated with liquidity pooling, which essentially is a function of price volatility.

The return from supplying liquidity is the ratio of fees over liquidity $r = \dfrac{\tau vol}{V}$. Given a certain volume of transactions, the return determines how much liquidity is provided in the pool. LPs accept to stake their wealth in exchange of a required return r^* which compensates them for the risk they take. In turn, the required return determines the amount of liquidity at the equilibrium: $V^* = \dfrac{\tau vol}{r^*}$. Figure 23 shows a simple representation of the equilibrium:

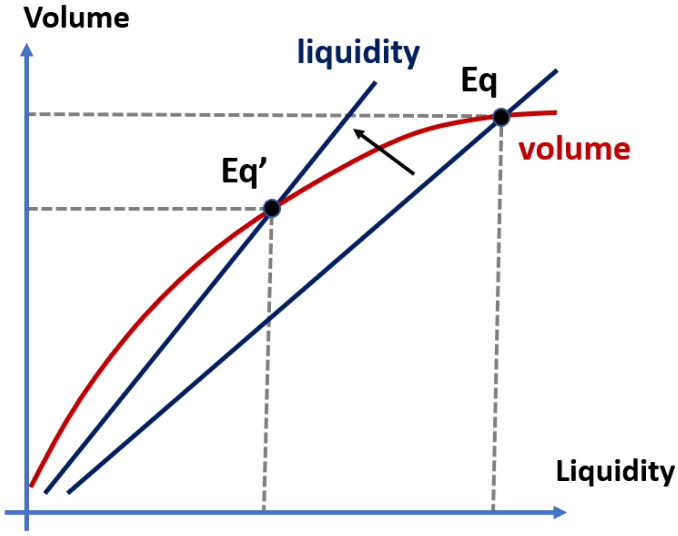
Figure 23: Volume meets liquidity at equilibrium



Liquidity (on the horizontal axis) is increasing with volume. The relation is assumed linear for simplicity. Volume (on the vertical axis) is also increasing with liquidity but at a diminishing rate, as additional liquidity reduces price slippage at a decreasing rate. The equilibrium is located at the intersection of the two curves.

Several scenarios can be studied from this baseline. First, suppose that perceived risk increases, due for instance to higher price volatility. In Figure 24, the equilibrium shifts from Eq to Eq' with less liquidity and less volume. For a given volume of transactions, less investors are willing to provide liquidity. The liquidity line shifts to the left. The liquidity shortage moves the return $r = \dfrac{\tau vol}{V}$ upward and reduces transaction volumes due to increased price slippage. The process continues until the return is high enough and compensates LPs for the

Figure 24: The effects of a change in perceived risk



additional risk.

Figure 25: Effects of a variation of the fee rate

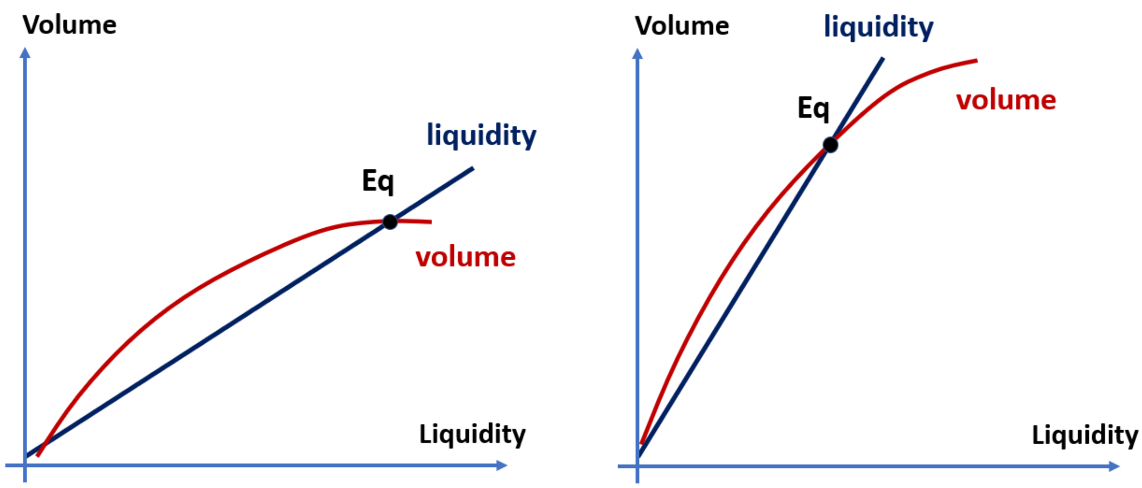


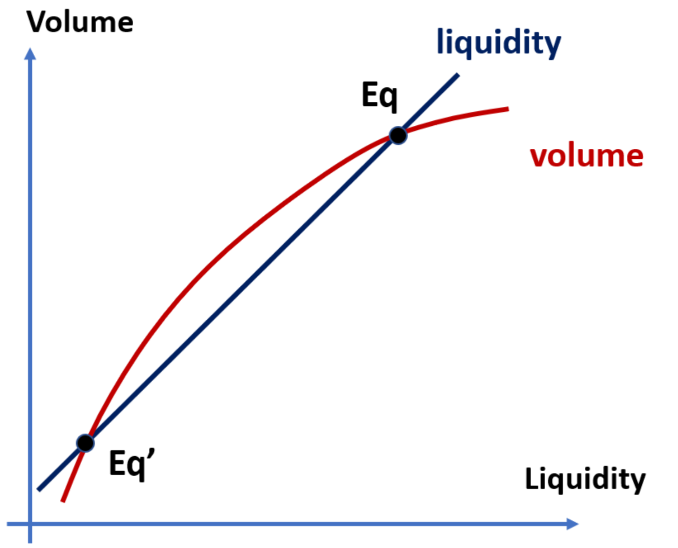
Figure 25 shows the effects of a change in fee rate τ. In the left-hand figure, the fee rate is raised whereas it is lowered in the right-hand figure. In the first case, volume is down and liquidity is up. Graphically, both curves have a smaller slope. There are less volume and more liquidity at the new equilibrium. The reduction of volume is attenuated by a smaller price slippage. Since by assumption the risk borne by LPs is unchanged, so is the rate of return they require for staking their wealth: $r^* = \frac{\tau vol}{V^*}$. Hence the liquidity increases up to the point at which the effect of τ on return is eliminated.

Conversely, if the fee rate is lowered, volumes are up and liquidity is down. Flying liquidity increases the return until it exactly compensates for lower fee rate.

Because liquidity chases volume and volume is attracted by low slippage costs, several equilibria are possible at the same time. In Figure 26, two equilibria exist: a high activity equilibrium in which high volume and abundant liquidity sustain each other and a low activity equilibrium in which low volume and liquidity coexist.

Moving the protocol from the low to the high activity equilibrium is tricky. Volume can be attracted by lowering the fee rate or liquidity can be bootstrapped by raising the fee rate but the two sides of the market cannot be moved to the same direction at the same time. One possibility, recently explored by Uniswap and its forked version Sushiswap, consists in incentivizing liquidity by distributing a governance token to LPs in addition to trade fees. The token has an intrinsic value as a small fraction of the fees is channeled toward the token holders.

Figure 26: Multiple equilibria



The operation has proved effective in bootstrapping both liquidity and volume. Whether high volume and transactions can be sustained in the long term by a regular distribution of tokens is still an open issue.

5 Conclusion

Constant product matching markets offer an innovative way for exchanging assets by allowing users to trade against a liquidity pools funded by LPs. By automating trading and price setting in smart contracts, the protocol avoid middlemen and economizes on platform costs. This allows LPs to earn sizable trade fees. Its functioning shows how an algorithm may trade with human counter-parties without knowing market prices. Trading under asymmetric information leads the protocol to forego a non-trivial fraction of the return when prices are volatile, which negatively impact the profitability for LPs. The missing return, commonly known as impermanent loss, is actually permanent if the counterfactual portfolio rebalances as the protocol does. Since an individual investor can rebalance at a better exchange rate than the protocol, the return delivered to LP will always lag behind the rebalanced portfolio, even if the price reverts to its initial value. In practice, LPs must strike a delicate balance between giving away returns to arbitrageurs and accumulating fees to achieve a satisfactory return.